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Contact Transformations and Conformal Group. I. Relativistic Theory

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Abstract

We find and identify the contact group of plane hyperbolic conformal geometry as a step to a better understanding of conformal invariance in physics.

1. *Introduction*

The conformal group, mainly in infinitesimal form, has recently aroused widespread interest in field theory. As in most physical theories, a better understanding of the mathematical foundation will illuminate such a theory. In this work we attempt to enlarge on this.

First we wish to remark that prior to the existing fairly acceptable interpretation of conformal transformations (Kastrup, 1968, 1966a, b), some authors tried to connect them with kinematical transformations (for example in Minkowski space (Fulton *et al.*, 1962a, b)); in fact the conformal relativistic group corresponds to the invariance group of reference systems with arbitrary, but constant, acceleration.

In spite of the fact that this interpretation has not been generally accepted (Kastrup, 1968), we will proceed from this point of view, hoping that the new mathematical features which emerge might be of use in the actual interpretation of conformal transformations.

The condition to be satisfied for all systems with constant relative accelerations can be expressed geometrically as a differential equation involving the 'second' acceleration $b = d^3x/dt^3$; the invariance group of the equation $b' = 0$ will provide the kinematical group for this situation.

Hill (1945) proved that such a group is precisely the conformal group. We are going to find the *contact* group which leaves this equation invariant (a precise definition will be given later), restricting ourselves to (1 + 1) space-time dimensions for simplicity. We shall then identify the conformal group as a subgroup of this larger contact group.

In Section 2 we define the infinitesimal extended contact transformations in the plane; in Section 3 the Lie algebra of the contact group is obtained for the third-order equation; and in Section 4 we obtain no larger group, in the point transformation case, for the second-order equation which corresponds to the normal Lorentz invariance. Finally we identify the contact group with the conformal group in three (3 + 1) dimensions.

Generalisations to realistic dimensions or possible use in field theory are reserved for future work.

2. *Contact Transformations*

A manifold $\mathscr{V}$ of odd dimension = $2n + 1$ with a 1-form θ such $\theta \wedge (d\theta)^n \neq 0$ is called a (exact) *contact manifold* (Nomizu, 1966; Abraham, 1967); a contact transformation is a diffeomorphism in a contact manifold which preserves θ 'projectively', i.e. g is of contact if:

$$\theta^g = h_g \theta \tag{2.1}$$

where h_g is a function h: $\mathscr{V} \to \mathbb{R}$; in other words $\theta = 0 \Rightarrow \theta^g = 0$; in classical language, a contact transformation leaves invariant the Pfaff equation. Locally, one can always write (Abraham, 1967):

$$\theta = dy - \sum_{i=1}^{n} p_i \, dx_i \tag{2.2}$$

if $(x_1 \ldots x_n; p_1 \ldots p_n; y)$ are local coordinates. In $2n + 1 = 3$, one has

$$\theta = dy - p \, dx \tag{2.2$'$}$$

and the following simple interpretation

$$\theta = 0 \Rightarrow p = \frac{dy}{dx}; \qquad \theta^g = 0 \Rightarrow p^g = \frac{dy^g}{dx^g} \tag{2.3}$$

i.e. if $\mathscr{V} = \mathscr{V}_3$ is the projective tangent bundle of the plane with local coordinates (x, y, p), contact transformations retain this fibre bundle structure (Abraham, 1967). Sophus Lie introduced the concept of contact transformations (Berührungtransformationen) to solve differential equations (Lie, 1892, Part II; Campbell, 1970). The appellation 'contact' is due to the fact that if (e.g. in the plane) two curves are tangent they are still tangent when transformed, because the point of tangency and the slope of the common tangent make up a point in the contact manifold, and contact transformations are point transformations in the contact manifold.

Let x, $x_0 = ct$, be coordinates in $\mathscr{V} = \mathbb{R}^2$. A one-parameter group of transformations $\mathscr{G}'_\tau$ has a generator (vector field) X such that

$$X = \xi_0(x_0, x)\frac{\partial}{\partial x_0} + \xi(x_0, x)\frac{\partial}{\partial x} \tag{2.4}$$

where $\xi_0 = Xx_0$ and $\xi = Xx$, the infinitesimal transformation is

$$x_0 \to x_0' = x_0(\tau) = x_0 + t\xi_0(x_0, x);$$
$$y_0 \to y_0' = x \to x' = x(\tau) = x + t\xi(x_0, x)$$

Now in $\mathbb{R}^3$ coordinates (x, x_0, p) an infinitesimal contact transformation is a vector field Y with $\mathscr{G}_\tau$ group such:

$$\theta^\tau = h_\tau\theta \Rightarrow L_Y\theta = h_f\theta \tag{2.5}$$

where $L_Y\theta$ is the Lie derivative of the contact form (2.2). If

$$Y = \xi_0(x, x_0, p)\,\partial/\partial x_0 + \xi(x, x_0, p)\,\partial/\partial x + \pi(x, x_0, p)\,\partial/\partial p \tag{2.6}$$

is a general vector field in $\mathbb{R}^3$, it generates an infinitesimal contact transformation and verifies (2.5), which is equivalent to:

$$(\xi_0\,dp - \pi\,dx_0) - d(p\xi_0 - \xi) = h_f(dx - p\,dx_0)$$

If $\Phi(x, y, p) = p\xi_0 - \xi$, the latter equation amounts to:

$$\xi_0 = \frac{\partial\Phi}{\partial p}; \qquad \pi = -\left(\frac{\partial}{\partial x_0} + p\frac{\partial}{\partial x}\right)\phi$$

$$f = \frac{\partial\phi}{\partial x}; \qquad \xi = p\frac{\partial\phi}{\partial p} - \phi \tag{2.7}$$

The function $\phi = \phi(x, y, p)$ is called the *generating function* of the contact transformation (Lie, 1892; Campbell, 1970).

A point transformation in $\mathbb{R}^2$, i.e. an ordinary transformation, also gives rise to an (extended) transformation in the contact bundle, and is of contact character, but not of the most general type; these will be called extended point transformations. If (2.4) is the generator the extended generator is (Ince, 1956):

$$\tilde{X} = X + \left[\frac{\partial\xi}{\partial x_0} + \left(\frac{\partial\xi}{\partial x} - \frac{\partial\xi_0}{\partial x_0}\right)p - \frac{\partial\xi_0}{\partial x}p^2\right]\frac{\partial}{\partial p} \tag{2.8}$$

Similarly one can extend a contact transformation to make it operate in the higher order derivatives. We shall need the formulae up to third order. Calling

$$\begin{cases} q = d^2x/dx_0{}^2 = dp/dx_0; & dp = q\,dx_0 \\ r = \dfrac{d^3x}{dx_0{}^3} = \dfrac{dq}{dx_0}; & dq = r\,dx_0 \end{cases} \tag{2.9}$$

and writing

$$\begin{aligned} q' &= q + \tau\kappa(x, x_0, p) \\ r' &= r + \tau\rho(x, x_0, p) \end{aligned} \tag{2.10}$$

from $dp' = q'\,dx'_0 = d(p + \tau\pi)$

$$d\pi = q\,d\xi_0 + \kappa\,dx_0 \tag{2.11}$$

As $d(\pi - q\,d\xi_0) = d\pi - q\,d\xi_0 - \xi_0 r\,dx_0$, it remains

$$d(\pi - q\xi_0) = \kappa\,dx_0 - \xi_0 r\,dx_0 \tag{2.12}$$

Defining the total derivative

$$\frac{d}{dx_0} = \frac{\partial}{\partial x_0} + p\frac{\partial}{\partial x} + q\frac{\partial}{\partial p} + r\frac{\partial}{\partial q} \tag{2.13}$$

we obtain $\pi - q\xi_0 = -(\partial\phi/\partial x_0)$ because $\phi = \phi(x, x_0, p)$ only. Therefore

$$\kappa = -\frac{d^2\phi}{dx_0{}^2} \tag{2.14}$$

independent of higher derivatives. Similarly it can be seen that

$$\rho = -\frac{d^3\phi}{dx_0{}^3} \tag{2.15}$$

The explicit form of κ and ρ is rather long:

$$-\kappa = \left(X^2 + 2qX\frac{\partial}{\partial p} + q^2\frac{\partial^2}{\partial p^2} + q\frac{\partial}{\partial x}\right)\phi \tag{2.16}$$

$$\begin{aligned} -\rho = {} & \left(X^3 + 3qX^2\frac{\partial}{\partial p} + 3q^2X\frac{\partial^2}{\partial p^2} + q^3\frac{\partial^3}{\partial p^3} + 3qX\frac{\partial}{\partial x} + 3q^2\frac{\partial^2}{\partial x\,\partial p}\right)\phi \\ & + r\left(3q\frac{\partial^2}{\partial p^2} + 3X\frac{\partial}{\partial p} + \frac{\partial}{\partial x}\right)\phi \end{aligned} \tag{2.17}$$

where

$$X = \frac{\partial}{\partial x_0} + p\frac{\partial}{\partial x}$$

3. *Acceleration Invariance Group*

Consider a relativistic motion in $\mathbb{R}^2$: ($x_0 = ct$, x). If it has a uniform acceleration the path is a hyperbola, and transformations which preserve these types of hyperbolae are precisely the Poincaré (1 + 1) transformations. Now the relation between the second acceleration $b = d^3x/dt^3$ and the acceleration b' in the proper system of the moving point is well known in special relativity (Pauli, 1967); it is:

$$b' = \frac{b}{(c^2 - v^2)^2} + \frac{3va^2}{(c^2 - v^2)^3} \tag{3.1}$$

with $v = dx/dt$, $a = d^2x/dt^2$, $b = d^3x/dt^3$. The group referred to in Section 1 as the transformation group between systems with arbitrary constant relative acceleration is the invariance group of $b' = 0$ or

$$\frac{d^3x}{dx_0{}^3} = -\frac{3\left(\dfrac{dx}{dx_0}\right)\left(\dfrac{d^2x}{dx_0{}^2}\right)^2}{1 - \left(\dfrac{dx}{dx_0}\right)^2} \tag{3.2}$$

which is, of course, the differential equation of all equilateral hyperbolae on the plane, $(x - \alpha)^2 - (y - \beta)^2 = \gamma^2$, and the point invariance group of (3.2) is the conformal group of the pseudo-euclidean plane.

To obtain the contact invariance group of (3.2) we write the equation in the new variables $x_0, x, p, q, r \to x'_0, x', p', a', r'$, which, under the group, are in accordance with (2.10) and similar expressions resulting in

$$\pi \frac{6p^2q^2}{1 - p^2} + (1 - p^2)\rho + 3q^2\pi + \sigma pq\kappa = 0 \tag{3.3}$$

From the expression for π, κ and ρ obtained the following system of partial differential equations results:

$$(1 - p^2)\frac{\partial^3\phi}{\partial p^3} = 3p\frac{\partial^2\phi}{\partial p^2} \tag{3.4}$$

$$(1 - p^2)^2\left[X\frac{\partial^2\phi}{\partial p^2} + \frac{\partial^2\phi}{\partial x\, \partial p}\right] + (1 + p^2)\left[\frac{\partial\phi}{\partial x_0} + p\frac{\partial\phi}{\partial x}\right] + p(1 - p^2)\left[X\frac{\partial\phi}{\partial p} + \frac{\partial\phi}{\partial x}\right] = 0 \tag{3.5}$$

$$(1 - p^2)\left[X^2\frac{\partial\phi}{\partial p} + X\frac{\partial\phi}{\partial x}\right] + 2pX^2\phi = 0 \tag{3.6}$$

$$X^3\phi = 0 \tag{3.7}$$

From (3.4)

$$\phi = -A\sqrt{(1-p^2)} + Bp + C \tag{3.8}$$

with A, B, C depending only on (x, x_0). Now (3.5)-(3.7) impose on A, B, C the restrictions

$$A_{x_0x_0} = -A_{xx}; \qquad A_{xx_0} = 0 \tag{3.9}$$

$$C_{x_0} = -B_x; \qquad C_x = -B_{x_0} \tag{3.10}$$

$$\begin{cases} B_{x_0x} = -C_{xx}; \qquad B_{x_0x_0} = -C_{x_0x} \\ 2B_{xx} + B_{x_0x_0} + 3C_{x_0x} = 0 \\ 2B_{x_0x_0} + C_{xx} + 3B_{x_0x} = 0 \end{cases} \tag{3.11}$$

'All third-order derivatives of A, B, C are zero' (3.12)

With this ϕ depends on ten arbitrary integration constants, and can be written on the form:

$$\begin{aligned}\phi = {} & \alpha_1(x_0^2 - x^2)\sqrt{(1-p^2)} + \alpha_2 x_0\sqrt{(1-p^2)} + \alpha_3 x\sqrt{(1-p^2)} \\ & + \alpha_4\sqrt{(1-p^2)} + \beta_1[p(x_0^2 + x^2)/2 - xx_0] \\ & + \beta_2[pxx_0 - \tfrac{1}{2}(x_0^2 + x^2) + \beta_3(px_0 - x) + \beta_4(px - x_0) + \beta_5 p + \beta_6]\end{aligned} \tag{3.13}$$

From (2.7) the most general generators can be written as:

$$X = \sum_{i=1}^{4} \alpha_i X_i + \sum_{j=1}^{6} \beta_j X_{j+4} \tag{3.14}$$

Where

$$\begin{aligned} X_1 &= \frac{p(x_0^2 - x^2)}{2\sqrt{(1-p^2)}}\frac{\partial}{\partial x_0} + \frac{x_0^2 - x^2}{2\sqrt{(1-p^2)}}\frac{\partial}{\partial x} + \sqrt{(1-p^2)}(x_0 - x_p)\frac{\partial}{\partial p} \\ X_2 &= \frac{px_0}{\sqrt{(1-p^2)}}\frac{\partial}{\partial x_0} + \frac{x_0}{\sqrt{(1-p^2)}}\frac{\partial}{\partial x} + \sqrt{(1-p^2)}\frac{\partial}{\partial p} \\ X_3 &= \frac{px}{\sqrt{(1-p^2)}}\frac{\partial}{\partial x_0} + \frac{x}{\sqrt{(1-p^2)}}\frac{\partial}{\partial x} + p\sqrt{(1-p^2)}\frac{\partial}{\partial p} \\ X_4 &= \frac{p}{\sqrt{(1-p^2)}}\frac{\partial}{\partial x_0} + \frac{1}{\sqrt{(1-p^2)}}\frac{\partial}{\partial x} \end{aligned} \tag{3.15}$$

and

$$X_5 = \frac{x_0^2 + x^2}{2}\frac{\partial}{\partial x_0} + x_0 x\frac{\partial}{\partial x} + x(1-p^2)\frac{\partial}{\partial p}$$

$$X_6 = xx_0 \frac{\partial}{\partial x_0} + \tfrac{1}{2}(x_0^2 + x^2)\frac{\partial}{\partial x} + x_0(1-p^2)\frac{\partial}{\partial p}$$

$$X_7 = x\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x} + (1-p^2)\frac{\partial}{\partial p}$$

$$X_8 = x_0\frac{\partial}{\partial x_0} + x\frac{\partial}{\partial x}$$

$$X_9 = \frac{\partial}{\partial x_0}$$

$$X_{10} = \frac{\partial}{\partial x} \tag{3.16}$$

The commutation relations can be obtained at once with the definitions:

$$\begin{aligned} &J_{41} = 2^{-1/2}(X_9 - X_5); && J_{42} = -2^{-1/2}(X_6 + X_{10}) \\ &J_{31} = 2^{-1/2}(X_9 + X_5); && J_{32} = 2^{-1/2}(X_6 - X_{10}) \\ &J_{12} = X_7; && J_{34} = -X_8 \\ &J_{51} = -X_2; && J_{52} = -X_3 \\ &J_{53} = \alpha^{-1/2}(X_4 - X_1); && J_{54} = 2^{-1/2}(X_4 + X_1) \end{aligned} \tag{3.17}$$

Taking the form

$$[J_{\alpha\beta}, J_{\gamma\delta}] = +(g_{\beta\gamma}J_{\alpha\delta} + g_{\alpha\delta}J_{\beta\gamma} - g_{\alpha\gamma}J_{\beta\delta} - g_{\beta\delta}J_{\alpha\gamma}) \tag{3.18}$$

(with diag g: $+ - + - +$) characteristic of the Lie algebra of the pseudo-orthogonal five-dimensional group $O_{3+2}(\mathbb{R})$.

The generators of the point group are easily identified, because ξ_0 and ξ of (2.7) must not be p-dependents. They are $X_5, \ldots X_{10}$, and generate the ordinary conformal group of the pseudo-euclidean plane: Γ_{11}, isomorphic to $O_{2,2}$. That $X_5, \ldots X_{10}$ close under commutation, i.e. $\Gamma_{1,1} \subset O_{3,2}$ as subgroup, is easy to prove.

For the euclidean case, see Campbell (1970) and Lie (1892, Part III). Hill (1945) also gives a short discussion in the Minkowskian case.

4. *Contact Group of Special Relativity*

As we have remarked the point invariance group of a relativistic uniformly accelerated motion

$$\frac{d^2x}{dt^2} : \left(c^2 - \left(\frac{dx}{dt}\right)^2\right)^2 = a_0 = \text{const.} \tag{4.1}$$

is the Poincaré group. Let us look for the contact group of this equation. In the notation of Section 3 equation (4.1) becomes:

$$\frac{q}{(1-p^2)^2} = a_0 c^4 = a_1 = \text{const.} \tag{4.2}$$

with

$$q = \frac{d^2x}{dx_0^2} = \frac{dp}{dx_0}$$

A method similar to the one used in Section 3, leads to

$$\kappa = -4a_1 p\pi(1-p^2) \tag{4.3}$$

and from the expressions for κ and π we obtain a new partial system; one of the equations is

$$\frac{\partial^2\phi}{\partial p^2} = 0 \tag{4.4}$$

which, combined with $\phi = p\xi_0 - \xi$, imply that $\partial_p \xi_0 = \partial_p \xi = 0$, i.e. we obtain $x_0 \to x_0 + \tau\xi_0(x, x_0)$; $x \to x + \tau\xi(x_0, x)$, which is an extended *point* transformation. The contact group of (4.1) is the point group, namely the inhomogeneous (1 + 1) Lorentz group. Perhaps this is the reason why no one has followed this scheme up to now.

Finally, the conformal contact group in (1 + 1) coincides with the (point) conformal group in (2 + 1) (which is $O_{3,2}$ (Dieudonné, 1971)). In a later paper we will try to show this equivalence more clearly.

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